

ON THE REGULAR SUM-FREE SETS*

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ABSTRACT. Cameron introduced a bijection between the set of sum-free sets and the set of all zero-one sequences. In this paper, we study the sum-free sets of natural numbers corresponding to certain zero-one sequences which contain the Cantor-like sequences and some substitution sequences, etc. Those sum-free sets considered as integer sequences are 2-regular. We also prove that sequences corresponding to certain sum-free sets are automatic.

1. INTRODUCTION

A set of integers, denoted by S , is called a *sum-free set* if $S \cap (S + S) = \emptyset$, where $S + S$ denotes the set of pairwise sums, i.e., $S + S = \{x + y : x, y \in S\}$. Thus, for any sum-free set S , there do not exist $x, y, z \in S$ for which $x + y = z$. It is natural to arrange the elements of a sum-free set S in ascending order, so $S = (S_n)_{n \geq 0}$ can also be treated as an integer sequence.

Cameron and Erdős [11, 12] studied the number of sum-free sets which are contained in the first n integers. They showed that the number of sum-free subsets of $\{\frac{n}{3}, \frac{n}{3} + 1, \dots, n\}$ is $O(2^{\frac{n}{3}})$. Calkin [4] showed that the Hausdorff dimension of the sum-free sets is at most 0.599. Calkin [5] proved that the number of the sum-free subsets of $\{1, 2, \dots, n\}$ is $o(2^{n(1/2+\epsilon)})$ for every $\epsilon > 0$. Łuczak and Schoen [15] studied the properties of k -sum-free sets with precise upper density.

There are several methods to construct infinite sum-free sets. One of them is to construct such a set directly using its definition; that is, to construct it from numbers which are not the sum of two earlier numbers. Another way is to construct the sum-free set from an infinite zero-one sequence. This method was introduced by Cameron. Cameron [9] defined a natural bijection between Σ and \mathfrak{S} , where Σ and \mathfrak{S} denote the set of all zero-one sequences and the set of all sum-free sets respectively.

1.1. The bijection between Σ and \mathfrak{S} . Let $S \in \mathfrak{S}$ be a sum-free set and \mathbf{v} be a ternary sequence defined as follow:

$$v_n = \begin{cases} 1, & \text{if } n \in S; \\ *, & \text{if } n \in S + S; \\ 0, & \text{otherwise.} \end{cases} \quad (1.1)$$

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By deleting all the $*$'s in \mathbf{v} , we obtain a unique zero-one sequence \mathbf{v}' . This process introduces a mapping from \mathfrak{S} to Σ . Calkin [6] showed that this mapping is a bijection and denoted its inverse by θ . Thus $\theta : \mathbf{v}' \mapsto S$ is a bijection from Σ to \mathfrak{S} .

Since there is a one-to-one correspondence between the set of zero-one sequences and the set of sum-free sets (of positive integers), one would like to know the connection between these two.

1.2. Periodicity of S . A sum-free set S is said to be *ultimately periodic* if there exist positive integers m, n_0 such that for all $n > n_0$, $n \in S$ if and only if $n+m \in S$. If $n_0 = 0$, then S is *periodic*. It is not easy to show whether a given sum-free set is periodic or not.

In [10], Cameron observed that if a sum-free set is (ultimately) periodic, the corresponding zero-one sequence is also (ultimately) periodic. This was proved by Calkin and Finch in [6]. Conversely, Cameron [6] also asked whether sum-free sets corresponding to (ultimately) periodic zero-one sequences are (ultimately) periodic or not. This question is still open. With the help of a computer, Calkin and Finch [6] presented some sum-free sets, which correspond to periodic zero-one sequences, and appear to be aperiodic (aperiodicity checked up to 10^7). There is no proof to show whether such sum-free sets are periodic or aperiodic. Calkin and Erdős [7] showed that a class of aperiodic sum-free sets S is incomplete, i.e., $\mathbb{N} \setminus (S + S)$ is an infinite set. Later, Calkin, Finch and Flowers [8] introduced the concept of difference density which can be used to test whether specific sets are periodic or not. These tests produced further evidence that certain sets are not ultimately periodic. Payne [16] studied the properties of some sum-free sets over an additive group.

1.3. Main results. Motivated by Cameron's question, in this paper, we will investigate the regularity of S . We would like to know what kind of zero-one sequences will lead to k -regular sum-free sets?

Definition 1. A sum-free set S is called a k -regular sum-free set if the sequence $(S_n)_{n \geq 0}$ is k -regular¹ for some $k \in \mathbb{N}$.

Let $\mathbf{c} = c_0 c_1 c_2 \cdots$ be a zero-one sequence with $c_0 = 1$. Let μ_n denote the number of zeros between the n -th and the $(n+1)$ -th ones in \mathbf{c} . An equivalent definition of μ_n is in Definition 2. The sum-free set corresponding to \mathbf{c} is denoted by S . Our results are stated as follows:

Main results I. Suppose the integer sequence $(\mu_n)_{n \geq 1}$ is 2-regular and satisfies the conditions

$$\begin{cases} \mu_{2^m} > \sum_{i=1}^{2^m-1} \mu_i + 2^m + \frac{3^m-1}{2}, \\ \mu_{2^m+k} = \mu_k, \quad \forall 0 < k < 2^m, \end{cases} \quad (1.2)$$

for all $m \geq 1$, then S is a 2-regular sum-free set.

For any fixed $l_1 \geq 0$, $l_2 \geq 0$ and $l_3 \geq 3$, let $\sigma(l_1, l_2, l_3)$ be a substitution over the alphabet $\{0, 1\}$ given by

$$1 \mapsto 1 \overbrace{0 \cdots 0}^{l_1} 1 \overbrace{0 \cdots 0}^{l_2}, \quad 0 \mapsto \overbrace{0 \cdots 0}^{l_3}. \quad (1.3)$$

¹For the definition of k -regular sequences, refer to Definition 3 in Section 3.1.

The sequence

$$\begin{aligned} \mathbf{c}_{l_1, l_2, l_3} &= \sigma(l_1, l_2, l_3)^\infty(1) \\ &= 10 \cdots 010 \cdots 010 \cdots 01 \cdots \end{aligned} \quad (1.4)$$

is called a *Cantor-like sequence*. In particular, $\mathbf{c}_{1,0,3}$ is the Cantor sequence. In the rest of this paper, since l_1, l_2, l_3 are fixed in the context, we will simply use \mathbf{c} to denote such Cantor-like sequences. It is worth pointing out that Cantor-like sequences contain a class of automatic sequences and a class of non-automatic substitution sequences.

Main results II. *Let \mathbf{c} be a Cantor-like sequence satisfying the assumption in Main results I. Then the corresponding sum-free set S modulo 2 is the Thue-Morse sequence² beginning by 1 up to a coding.*

We also investigate the zero-one sequences corresponding to a class of sum-free sets generated from the set $(bn + 1)_{n \geq 0}$ through base changing where $b \in \mathbb{N}$ and $b \geq 2$. For example, let $S = \{1, 3, 5, 7, \dots\}$ where $S_n = (2n + 1)_{n \geq 0}$. Suppose $n = \sum_{i=1}^k n_i \cdot 2^{i-1}$, with $n_i \in \{0, 1\}$ for any $1 \leq i \leq k$, denote $S'_n := \sum_{i=1}^k n_i \cdot 3^i + 1$. Then $S' = \{1, 4, 10, 13, 28, 31, 37, 40, \dots\}$ is also sum-free. Moreover, the corresponding zero-one sequence of S' is the Cantor sequence (see Remark 6).

Main results III. *Suppose $b \geq 2$, let S be the sum-free set given by $S_n = \sum_{i=1}^k n_i (2b - 1)^i + 1$ where $n = \sum_{i=1}^k n_i 2^{i-1}$ with $n_i \in \{0, 1\}$, then $\theta^{-1}(S)$ is an automatic sequence.*

This paper is organized as follows. In Section 2, we introduce some definitions and auxiliary lemmas which are useful in the proof of Theorem 1. In Section 3, we give the explicit value of the sum-free set S corresponding to \mathbf{c} and discuss the regularity of S . A class of sum-free sets S corresponding to Cantor-like sequences is investigated in Section 4. In the last section, we discuss the automaticity of zero-one sequences corresponding to certain sum-free sets.

2. PRELIMINARY

2.1. m -complement of nonnegative integers. For $b \in \mathbb{N}$, let $\Sigma_b := \{0, 1, \dots, b-1\}$. Let Σ_b^k be the set of words of length k over Σ_b and $\Sigma_b^* = \bigcup_{k \geq 0} \Sigma_b^k$. The b -ary expansion of n is denoted by

$$(n)_b := \epsilon_k \epsilon_{k-1} \cdots \epsilon_1 \in \Sigma_b^k,$$

where $\epsilon_k \neq 0$. The b -ary number of $\mathbf{w} = w_k w_{k-1} \cdots w_1 \in \Sigma_b^k$ is given by

$$[\mathbf{w}]_b := \sum_{i=1}^k w_i \cdot b^{i-1}.$$

For $\mathbf{w} = w_k w_{k-1} \cdots w_1 \in \Sigma_b^k$, the *complement* of \mathbf{w} is the word

$$\bar{\mathbf{w}} = \bar{w}_k \bar{w}_{k-1} \cdots \bar{w}_1,$$

where $\bar{\epsilon} = 1 - \epsilon$ for any $\epsilon \in \Sigma_b$.

²The Thue-Morse sequence beginning by 1 is the infinite sequence $\mathbf{t} = t_0 t_1 \cdots \in \{0, 1\}^{\mathbb{N}}$ satisfying the following relations: $t_0 = 1, t_{2n} = t_n, t_{2n+1} = 1 - t_n$.

Let $m \in \mathbb{N}$ be fixed and $n = \epsilon_m 2^{m-1} + \dots + \epsilon_1 2^0 < 2^m$ with $\epsilon_i \in \Sigma_2$. In this case, $\epsilon_m \epsilon_{m-1} \dots \epsilon_1$ is a 2-ary representation of n of length m which allows leading zeros, denoted by $(n)_{2,m}$. The m -complement of n , denoted by \bar{n} , is given by

$$\bar{n} = \left[\overline{(n)_{2,m}} \right]_2 = \bar{\epsilon}_m 2^{m-1} + \dots + \bar{\epsilon}_1 2^0.$$

For example, when $m = 3$, then $\bar{1} = 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 = 6$, $\bar{3} = 2^2 = 4$. We also use notation ‘ $0n$ ’ and ‘ $1n$ ’ in form, which denote integers of 2-ary representation ‘ $0\epsilon_m \epsilon_{m-1} \dots \epsilon_1$ ’ and ‘ $1\epsilon_m \epsilon_{m-1} \dots \epsilon_1$ ’ respectively. In fact,

$$0n = n, \quad 1n = 2^m + n.$$

2.2. Two auxiliary bijections. Let $(h(i))_{i \geq 1}$ be a strictly increasing positive integer sequence. Define the mapping $f : \Sigma_2^m \times \Sigma_2^m \rightarrow \mathbb{N}$ by

$$(x, y) \mapsto f(x, y) = \sum_{i=1}^m (x_i + y_i) h(i) + 2.$$

Let \sim be the equivalence relation on $\Sigma_2^m \times \Sigma_2^m$ induced by f , i.e.,

$$(x, y) \sim (x', y') \text{ if and only if } f(x, y) = f(x', y').$$

Denote $\tilde{\Sigma}^m := (\Sigma_2^m \times \Sigma_2^m) / \sim$ for simplicity. Hence, we have defined an injection

$$f : \tilde{\Sigma}^m \rightarrow \mathbb{N}.$$

For any $m \geq 0$, define two mappings φ_m, ψ_m as follows: for any $(u, v) \in \tilde{\Sigma}^m$,

$$\begin{aligned} (u, v) &\mapsto \varphi_m(u, v) = (\bar{u}, \bar{v}), \\ (u, v) &\mapsto \psi_m(u, v) = (0u, 1v). \end{aligned}$$

Notice that

$$\begin{aligned} f \circ \varphi_m(u, v) &= f(\bar{u}, \bar{v}) \\ &= \sum_{i=1}^m (\bar{u}_i + \bar{v}_i) h(i) + 2 \\ &= 2 \sum_{i=1}^m h(i) - f(u, v) + 4, \end{aligned} \tag{2.1}$$

and

$$f \circ \psi_m(u, v) = f(0u, 1v) = f(u, v) + h(m+1). \tag{2.2}$$

By (2.1) and (2.2),

$$(\bar{u}, \bar{v}) \sim (\bar{u}', \bar{v}') \iff (u, v) \sim (u', v') \iff (0u, 1v) \sim (0u', 1v').$$

Thus φ_m and ψ_m are well-defined injections.

But these two mappings are not always bijections. Additional conditions are needed to ensure that. Before this, we need the following notation: for any $0 \leq k < 2^m$ with $k = \epsilon_m 2^{m-1} + \dots + \epsilon_1 2^0$ and $\epsilon_i \in \Sigma_2$, denote

$$M_k := \sum_{i=1}^m \epsilon_i h(i) + 1, \quad M_{2^m+k} := M_k + h(m+1). \tag{2.3}$$

Note that $M_{2^m-1} = \sum_{i=1}^m h(i) + 1$. Moreover, for any $m \geq 0, 0 \leq k < 2^m$, let

$$\begin{aligned} L_m &:= \{(u, v) \in \tilde{\Sigma}^m : f(u, v) \leq M_{2^m-1}\}, \\ R_m &:= \{(u, v) \in \tilde{\Sigma}^m : M_{2^m-1} + 1 < f(u, v) \leq 2M_{2^m-1}\}, \\ L(k) &:= \{(u, v) \in \tilde{\Sigma}^m : M_k < f(u, v) < M_k + K\}, \\ R(k) &:= \{(0u, 1v) \in \tilde{\Sigma}^{m+1} : M_{2^m+k} < f(0u, 1v) < M_{2^m+k} + K\}, \end{aligned}$$

where $K = \mu_k + \alpha_k + 1$, and $(\mu_i)_{i \geq 0}, (\alpha_i)_{i \geq 0}$ are positive integer sequences.

Lemma 1. $\varphi_m|_{L_m}$ is a bijection from L_m to R_m .

Proof. By (2.1), $f(\bar{u}, \bar{v}) = 2M_{2^m-1} - f(u, v) + 2$. Hence, if $(u, v) \in L_m$, then

$$M_{2^m-1} + 2 \leq f(\bar{u}, \bar{v}) \leq 2M_{2^m-1},$$

which implies $\varphi_m(u, v) = (\bar{u}, \bar{v}) \in R_m$.

Note that $\varphi_m(\bar{u}, \bar{v}) = (u, v)$. To show φ_m is a surjection, we only need to show $(\bar{u}, \bar{v}) \in L_m$ for any $(u, v) \in R_m$. In fact, if $(u, v) \in R_m$, then

$$f(\bar{u}, \bar{v}) = 2M_{2^m-1} - f(u, v) + 2 < M_{2^m-1} + 1.$$

□

Lemma 2. For any $0 \leq k < 2^m$, $\psi_m|_{L(k)}$ is a bijection from $L(k)$ to $R(k)$.

Proof. Note that $M_{2^m+k} = h(m+1) + M_k$. By (2.2), $(u, v) \in L(k)$ if and only if

$$M_{2^m+k} < f(0u, 1v) = f(u, v) + h(m+1) < M_{2^m+k} + K.$$

Thus $\psi_m|_{L(k)}$ is a bijection. □

Remark 1. By Lemma 1 and Lemma 2, for $0 \leq k < 2^m$,

$$\text{Card } L_m = \text{Card } R_m, \quad \text{Card } L(k) = \text{Card } R(k).$$

3. REGULARITY OF S

Let \mathbf{w} be a zero-one sequence beginning by 1, and $S = \theta(\mathbf{w})$ be the corresponding sum-free set. In this section, we will characterize S using the properties of $S + S$.

3.1. Elementary observations of S . Now assume that $\mathbf{v} = (v_n)_{n \geq 0}$ is the sequence generated from S by (1.1). In fact, \mathbf{v} labels all the natural numbers by 0, 1 or *. And only those integers belonging to S are labeled by 1. When we just care about the number of integers labeled by 0 or *, we will use “the number of 0’s (or *’s)” rather than “the number of integers labeled by 0’s (or *’s)”. This abuse of language does not cause any ambiguity in the paper.

Definition 2. For any $n \geq 1$, denote by μ_n (resp. α_n), the number of 0’s (resp. *’s) between S_{n-1} and S_n . In detail,

$$\begin{aligned} \mu_n &:= \text{Card } \{i \in \mathbb{N} : v_i = 0, S_{n-1} < i < S_n\}; \\ \alpha_n &:= \text{Card } \{i \in \mathbb{N} : v_i = *, S_{n-1} < i < S_n\}. \end{aligned}$$

Remark 2. (1) For any $n \geq 1$,

$$S_n - S_{n-1} = \mu_n + \alpha_n + 1.$$

Thus to study S , we need to study the properties of $(\mu_n)_{n \geq 1}$ and $(\alpha_n)_{n \geq 1}$.

(2) Since \mathbf{v} converts to \mathbf{w} by deleting all the $*$'s, μ_n represents the number of 0's between the n -th and $(n+1)$ -th '1' in \mathbf{w} , i.e.,

$$\mathbf{w} = 1 \overbrace{0 \cdots 0}^{\mu_1} 1 \overbrace{0 \cdots 0}^{\mu_2} 1 \overbrace{0 \cdots 0}^{\mu_3} 1 \cdots.$$

During the proof of Theorem 1, we need the following two notation: for any $n \geq 1$,

$$g(n) := \mu_n + \alpha_n, \quad (3.1)$$

$$h(n) := \left(\sum_{i=1}^{2^{n-1}} g(i) \right) + 2^{n-1}. \quad (3.2)$$

Thus $S_n = S_{n-1} + g(n) + 1$ and $h(n)$ is strictly increasing.

Lemma 3. Suppose $g(2^k + i) = g(i)$ for $0 \leq k < m$, $0 < i < 2^k$. Then

$$h(m+1) = \sum_{i=1}^m h(i) + g(2^m) + 1.$$

Proof. By (3.2),

$$\begin{aligned} & h(k+1) - h(k) \\ &= \left(\sum_{i=1}^{2^k} g(i) + 2^k \right) - h(k) \\ &= \left(\sum_{i=1}^{2^{k-1}-1} (g(i) + g(2^{k-1} + i)) + g(2^{k-1}) + g(2^k) + 2^k \right) - h(k) \\ &= h(k) + g(2^k) - g(2^{k-1}). \end{aligned}$$

Add the last equation up from $k = 1$ to m . The result follows. \square

Theorem 1. Let $\mathbf{c} := 1 \overbrace{0 \cdots 0}^{\mu_1} 1 \overbrace{0 \cdots 0}^{\mu_2} 1 \cdots$. Denote its corresponding sum-free set by $S = (S_n)_{n \geq 0}$. If the sequence $(\mu_n)_{n \geq 1}$ satisfies (1.2), then for every integer $n \geq 1$, we have

(1) if $n = 2^k(2j+1)$ for some $k, j \geq 0$, then

$$\alpha_n = \frac{3^k + 1}{2}; \quad (3.3)$$

(2) if $(n)_2 = \epsilon_m \epsilon_{m-1} \cdots \epsilon_1$, then

$$S_n = 1 + \sum_{i=1}^m \epsilon_i h(i).$$

Proof of Theorem 1. We prove this theorem by induction on n .

Step1: Since $\alpha_1 = 1$, $S_1 = \mu_1 + 3$, the conclusion is true for $n = 1$.

Step2: Assume the result is true for $n < 2^m$; that is, if $n = 2^k(2j+1)$, then $\alpha_n = \frac{3^k+1}{2}$. Thus $\alpha_{2^p+q} = \alpha_q, \forall 0 < q < 2^p, 0 < p < m$. Moreover, if $(n)_2 = \epsilon_m \epsilon_{m-1} \cdots \epsilon_1$, then $S_n = 1 + \sum_{i=1}^m \epsilon_i h(i)$.

Hence, it suffices to show that the result is also true for $2^m \leq n < 2^{m+1}$.

Step2.1 Let $n = 2^m$. By the induction hypothesis and (3.3), we have

$$\begin{aligned}
\sum_{i=1}^{2^m-1} \alpha_i &= \sum_{i=1}^{2^{m-1}-1} \alpha_i + \alpha_{2^{m-1}} + \sum_{i=1}^{2^{m-1}-1} \alpha_{2^{m-1}+i} = 2 \sum_{i=1}^{2^{m-1}-1} \alpha_i + \alpha_{2^{m-1}} \\
&= 2^2 \sum_{i=1}^{2^{m-2}-1} \alpha_i + 2\alpha_{2^{m-2}} + \alpha_{2^{m-1}} \cdots = \sum_{i=1}^m 2^{i-1} \alpha_{2^{m-i}} \\
&= \frac{3^m - 1}{2}.
\end{aligned} \tag{3.4}$$

Now we will evaluate α_{2^m} . By (3.4) and (1.2),

$$\begin{aligned}
S_{2^m-1} &= S_0 + \sum_{i=1}^{2^m-1} (\mu_i + \alpha_i) + 2^m - 1 \\
&= \sum_{i=1}^{2^m-1} \mu_i + 2^m + \frac{3^m - 1}{2} < \mu_{2^m}.
\end{aligned}$$

Thus for $i, j \leq 2^m - 1$,

$$S_i + S_j \leq 2S_{2^m-1} < S_{2^m-1} + \mu_{2^m}. \tag{3.5}$$

Thus to evaluate α_{2^m} , we only need to count the number of $*$'s between S_{2^m-1} and $2S_{2^m-1}$. It is easy to see that $S_0 + S_{2^m-1} = S_{2^m-1} + 1$ is one of such $*$'s. Denote

$$\begin{aligned}
\tilde{L}_m &:= \{x \in S + S : S_0 < x < S_{2^m-1}\}, \\
\tilde{R}_m &:= \{x \in S + S : S_{2^m-1} + 1 < x < 2S_{2^m-1}\}.
\end{aligned}$$

Then

$$\alpha_{2^m} = \text{Card } \tilde{R}_m + 1.$$

Since f is an injection and $\tilde{L}_m = f(L_m)$, $\tilde{R}_m = f(R_m)$, by Lemma 1, we have

$$\text{Card } \tilde{L}_m = \text{Card } \tilde{R}_m.$$

Hence,

$$\begin{aligned}
\alpha_{2^m} &= \text{Card } \tilde{L}_m + 1 \\
&= \sum_{i=1}^m 2^{m-i} \alpha_{2^{i-1}} + 1 \\
&= \frac{3^m + 1}{2}.
\end{aligned}$$

Now we will give the expression of S_{2^m} . By (1.2) and (3.3), for $0 < j < m$ and $0 < k < 2^j$,

$$g(2^j + k) = \alpha_{2^j+k} + \mu_{2^j+k} = \alpha_k + \mu_k = g(k).$$

Let $(2^m)_2 = 1 \overbrace{0 \cdots 0}^m =: \epsilon_{m+1}(2^m) \epsilon_m(2^m) \cdots \epsilon_1(2^m)$. By Lemma 3 and the induction assumption, we have

$$\begin{aligned} S_{2^m} &= S_{2^m-1} + g(2^m) + 1 \\ &= \sum_{k=1}^m h(k) + g(2^m) + 2 \\ &= h(m+1) + 1 \\ &= 1 + \sum_{k=1}^{m+1} \epsilon_k(2^m) h(k). \end{aligned}$$

Step2.2: Now we prove the results for $2^m < n < 2^{m+1}$. Let $n = 2^m + k$ where $1 \leq k < 2^m$, and we prove the results by induction on k . For $1 \leq k < 2^m$, assume the result is true for $n < 2^m + k$. We will prove the results for $n = 2^m + k$.

Firstly, we need to determine the value of S_{2^m+k} . Denote $\Xi_{2^m+k-1} := \{x \in S : x \leq S_{2^m+k-1}\}$. Then

$$\Xi_{2^m+k-1} + \Xi_{2^m+k-1} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3,$$

where

$$\begin{aligned} \mathcal{S}_1 &:= \{S_i + S_j : i, j < 2^m\}, \\ \mathcal{S}_2 &:= \{S_i + S_j : i < 2^m, 2^m \leq j \leq 2^m + k - 1\}, \\ \mathcal{S}_3 &:= \{S_i + S_j : 2^m \leq i, j \leq 2^m + k - 1\}. \end{aligned}$$

In fact, if $i, j < 2^m$, then by (3.5) we have $S_i + S_j < S_{2^m}$. Thus $\max \mathcal{S}_1 < S_{2^m}$. Note that by the induction hypothesis, $S_{2^m+k-1} = h(m+1) + S_{k-1}$ and $S_{2^m} = h(m+1) + 1$. Since $(S_n)_{n \geq 1}$ is strictly increasing, we have for $i, j \geq 2^m$,

$$\begin{aligned} S_i + S_j &> S_k + S_{2^m-1} \\ &= (S_{k-1} + \mu_k + \alpha_k + 1) + (h(m+1) + 1) - 1 \\ &= S_{2^m+k-1} + \mu_k + \alpha_k + 1 =: s. \end{aligned}$$

Therefore $\min \mathcal{S}_3 > s$. Note that $s = h(m+1) + S_k$. Set $K = g(k) + 1$, then $S_k = S_{k-1} + K$ and $s = S_{2^m+k-1} + K$. Let

$$\begin{aligned} \tilde{L}(k) &:= \{x \in S + S : S_{k-1} < x < S_k\}, \\ \tilde{R}(k) &:= \{x \in S + S : S_{2^m+k-1} < x < s\}. \end{aligned}$$

Clearly $f(L(k)) = \tilde{L}(k)$ and $f(R(k)) \subset \tilde{R}(k)$. Now we will show that $f(R(k)) = \tilde{R}(k)$. Since f is an injection, by Lemma 2, we have

$$\text{Card } \tilde{R}(k) \geq \text{Card } \tilde{L}(k) = \alpha_k.$$

Suppose $S_{2^m+k} < s$. The way that we construct S from a zero-one sequence implies that there are μ_k 0's, at least α_k *'s and at least one 1 between S_{2^m+k-1} and s . However there are only $g(k)$ integers between S_{2^m+k-1} and s , which is a contradiction. Therefore $S_{2^m+k} \geq s$. In this case, $f(R(k)) = \tilde{R}(k)$. Hence

$$\text{Card } \tilde{R}(k) = \alpha_k.$$

And the number s must be labeled by $*$ or 1 , i.e., $v_s = *$ or 1 . Since $\max \mathcal{S}_1 < S_{2^m}$ and $\min \mathcal{S}_3 > s$, then $s \in \mathcal{S}_2$ if $v_s = *$. That is,

$$\exists i < 2^m, j \geq 2^m, \text{ s.t., } s = S_i + S_j.$$

Then there exist $i', j' < 2^m$ satisfying $i = 0i', j = 1j'$, and

$$s = S_k + h(m+1) = S_i + S_j = S_{i'} + S_{j'} + h(m+1).$$

Thus $S_k = S_{i'} + S_{j'}$ which contradicts the construction of S_k .

Therefore $v_s = 1$ and $s = S_{2^m+k}$. Combining this fact with (1.2), $\alpha_k = \alpha_{2^m+k}$. The proof is completed. \square

Remark 3. $(h(n))_{n \geq 1}$ is a numeration system (for detail, refer to [3, Chapter 3]. In fact, by Lemma 3, for any $j \geq 1$, $\sum_{i=1}^{j-1} h(i) < h(j)$.

Definition 3 (Allouche and Shallit [1]). A sequence $(t(n))_{n \geq 0}$ is called k -regular if there exist m subsequences of n , say $\{n_l^{(j)}\}_{l \geq 0}$ ($0 \leq j \leq m-1$), which satisfy for any $i \geq 0$ and $0 \leq b < k^i$, the subsequence $(t(k^i n + b))_{n \geq 0}$ is a \mathbb{Z} -linear combination of $t(n_l^{(j)})$.

Lemma 4 (Allouche and Shallit [1]). If $(u_n)_{n \geq 1}$ and $(v_n)_{n \geq 1}$ are both k -regular, then $(u_n + v_n)_{n \geq 1}$ and $(\sum_{i=1}^n u_i)_{n \geq 1}$ are also k -regular.

Theorem 2. If $(\mu_n)_{n \geq 1}$ is 2-regular and (1.2) holds, then the sequence $(S_n)_{n \geq 0}$ is 2-regular.

Proof. Since $(\mu_n)_{n \geq 1}$ is 2-regular, by Lemma 4, so is $(\sum_{i=1}^n \mu_i)_{n \geq 1}$. And (3.3) in Theorem 1 implies that

$$\begin{cases} \alpha_{2n} = 3\alpha_n - 1, \\ \alpha_{2n+1} = 1. \end{cases}$$

Thus $(\alpha_n)_{n \geq 1}$ is 2-regular. By Lemma 4, $(\sum_{i=1}^n \alpha_i)_{n \geq 1}$ is 2-regular. Note that $(n+1)_{n \geq 1}$ is also 2-regular, so by Lemma 4,

$$S_n = \sum_{i=1}^n \mu_i + \sum_{i=1}^n \alpha_i + (n+1)$$

is 2-regular. \square

4. EXAMPLES

Let Σ_2^* be the set of finite words over alphabet $\{0, 1\}$. For any $w \in \Sigma_2^*$, the length of w is denoted by $|w|$. Denote by $|w|_0$ and $|w|_1$ the number of 0's and 1's in w respectively.

The following lemma completely characterizes the gap between two adjacent 1's in the Cantor-like sequence \mathbf{c} .

Lemma 5. For any $l_1 \geq 0$, $l_2 \geq 0$, $l_3 \geq 3$ and all $n \geq 1$,

$$\mu_n = \frac{l_2(l_3^k - 1)}{l_3 - 1} + l_1 l_3^k \quad (4.1)$$

where $n = 2^k(2j+1)$ for some $k, j \geq 0$.

Proof. Let $\sigma := \sigma(l_1, l_2, l_3)$. It is easy to see that $|\sigma^m(1)|_1 = 2^m$ and $|\sigma^m(0)| = |\sigma^m(0)|_0 = l_3^m$ for any $m \geq 0$.

Now, we prove this result by induction on n . It is clear that $\mu_1 = l_1$. Assume that the result is true for all $n < 2^m$; we prove it for $2^m \leq n < 2^{m+1}$. Since $\sigma^\infty(1)$ begins with $\sigma^{m+1}(1)$ and $|\sigma^{m+1}(1)|_1 = 2^{m+1}$, in order to evaluate μ_n for $2^m \leq n < 2^{m+1}$, we only need to investigate $\sigma^{m+1}(1)$. Note that $\sigma^m(1)$ begins with '1' and

$$\begin{aligned}
\sigma^{m+1}(1) &= \sigma^m(\sigma(1)) = \sigma^m(\overbrace{10 \dots 0}^{l_1} \overbrace{10 \dots 0}^{l_2}) \\
&= \sigma^m(1) \overbrace{\sigma^m(0) \dots \sigma^m(0)}^{l_1} \sigma^m(1) \overbrace{\sigma^m(0) \dots \sigma^m(0)}^{l_2} \\
&= \sigma^{m-1}(\overbrace{10 \dots 0}^{l_1} \overbrace{10 \dots 0}^{l_2}) \overbrace{\sigma^m(0) \dots \sigma^m(0)}^{l_1} \sigma^m(1) \overbrace{\sigma^m(0) \dots \sigma^m(0)}^{l_2} \\
&= \sigma^{m-1}(1) \overbrace{\sigma^{m-1}(0) \dots \sigma^{m-1}(0)}^{l_1} \sigma^{m-1}(1) \overbrace{\sigma^{m-1}(0) \dots \sigma^{m-1}(0)}^{l_2} \\
&\quad \overbrace{\sigma^m(0) \dots \sigma^m(0)}^{l_1} \sigma^m(1) \overbrace{\sigma^m(0) \dots \sigma^m(0)}^{l_2}; \tag{4.2}
\end{aligned}$$

then we have

$$\mu_{2^m} = \sum_{i=0}^{m-1} l_2 l_3^i + l_1 l_3^m = \frac{l_2(l_3^m - 1)}{l_3 - 1} + l_1 l_3^m. \tag{4.3}$$

From (4.2) and the fact $|\sigma^{m+1}(1)|_1 = 2^{m+1}$, we know that the block of consecutive zeros between n -th and $(n+1)$ -th '1' will appear in the second $\sigma^m(1)$ of $\sigma^{m+1}(1)$ and for all $0 < i < 2^m$,

$$\mu_{2^m+i} = \mu_i. \tag{4.4}$$

Since for any $i = 2^k(2j+1)$ where $0 < k < m$, we have $2^m + i = 2^k(2j' + 1)$. Equation (4.4) implies that for all $0 < i < 2^m$,

$$\mu_{2^m+i} = \mu_i = \frac{l_2(l_3^k - 1)}{l_3 - 1} + l_1 l_3^k. \tag{4.5}$$

Therefore the result follows from (4.3) and (4.5). \square

Theorem 3. Let \mathbf{c} be the Cantor-like sequence satisfying $7l_3 \geq 4(l_1 + l_2) + 17$ and $l_1(l_3 - 1) + l_2 > 3$, and S be the sum-free set corresponding to \mathbf{c} . Then the sequence $(S_n)_{n \geq 0}$ is 2-regular.

Proof. According to Theorem 2, we only need to show that the gap sequence $(\mu_n)_{n \geq 1}$ of \mathbf{c} is 2-regular and satisfies (1.2). By Lemma 5, we have

$$\begin{cases} \mu_{2n} = l_3 \mu_n + l_2, \\ \mu_{2n+1} = l_1, \end{cases} \tag{4.6}$$

which implies $(\mu_n)_{n \geq 1}$ is 2-regular.

$$\begin{aligned} \sum_{i=1}^{2^m-1} \mu_i &= \sum_{i=0}^{2^{m-1}-1} \mu_{2i+1} + \sum_{i=1}^{2^{m-1}-1} \mu_{2i} \\ &= 2^{m-1}l_1 + l_3 \sum_{i=1}^{2^{m-1}-1} \mu_i + (2^{m-1} - 1)l_2. \end{aligned}$$
$$l_3 \sum_{i=1}^{2^{m-1}-1} \mu_i = \sum_{i=1}^{2^m-1} \mu_i - 2^{m-1}(l_1 + l_2) + l_2. \quad (4.7)$$
$$\begin{aligned}
\mu_{2^{m+1}} &= l_3 \mu_{2^m} + l_2 \\
&\geq l_3 \left(\sum_{i=1}^{2^m-1} \mu_i + 2^m + \frac{3^m-1}{2} + 1 \right) + l_2 \\
&= \sum_{i=1}^{2^{m+1}-1} \mu_i - 2^m(l_1 + l_2) + l_3(2^m + \frac{3^m+1}{2}) + 2l_2. \tag{4.8}
\end{aligned}$$
$$\begin{aligned}
& 2^m(l_3 - l_1 - l_2) + l_3 \frac{3^m + 1}{2} \\
> 2^m \left(2 - \frac{3(l_3 - 3)}{4} \right) + l_3 \frac{3^m}{2} \\
\geq 2^{m+1} + 3^m \left(\frac{l_3}{2} - \frac{3}{4} \left(\frac{2}{3} \right)^m (l_3 - 3) \right) \\
\geq 2^{m+1} + 3^m \left(\frac{l_3}{2} - \frac{1}{2} (l_3 - 3) \right) = 2^{m+1} + \frac{3^{m+1}}{2}. \tag{4.9}
\end{aligned}$$
[illegible]

is

$$S = \{1, 3, 15, 17, 69, 71, 83, 85, 333, 337, 349, 353, 415, 417, 431, 435, \dots\}.$$

According to Theorem 3, S is 2-regular.

From Theorem 3, we have the following two results. In the following, the symbol \equiv means equality modulo 2.

Corollary 1. *Let $(S_n)_{n \geq 0}$ be the sum-free set in Theorem 3. For $0 \leq j \leq 3$, the subsequence $(S_{4n+j})_{n \geq 0}$ is either periodic or the Thue-Morse sequence up to a coding.*

Proof. Recall that by (3.4), we have

$$\sum_{i=1}^{2^n-1} \alpha_i = \frac{3^n - 1}{2}.$$

Using (4.7) several times, we have

$$\begin{aligned} \sum_{i=1}^{2^n-1} \mu_i &= l_3 \sum_{i=1}^{2^{n-1}-1} \mu_i + [2^{n-1}(l_1 + l_2) - l_2] \\ &= l_3^2 \sum_{i=1}^{2^{n-2}-1} \mu_i + l_3[2^{n-2}(l_1 + l_2) - l_2] + [2^{n-1}(l_1 + l_2) - l_2] \\ &= \dots \\ &= \sum_{i=1}^n l_3^{i-1} [2^{n-i}(l_1 + l_2) - l_2] \\ &= \frac{l_1 + l_2}{l_3 - 2} (l_3^n - 2^n) - l_2 \frac{l_3^n - 1}{l_3 - 1}. \end{aligned}$$

By (3.3), (4.1) and the previous two equations, we have

$$\begin{aligned} \sum_{i=1}^{2^n-1} (\mu_i + \alpha_i) &= \sum_{i=1}^{2^{n-1}-1} (\mu_i + \alpha_i) + \mu_{2^{n-1}} + \alpha_{2^{n-1}} \\ &= \frac{l_1 + l_2}{l_3 - 2} (l_3^{n-1} - 2^{n-1}) - l_2 \frac{l_3^{n-1} - 1}{l_3 - 1} + \frac{3^{n-1} - 1}{2} \\ &\quad + (\mu_{2^{n-1}} + \alpha_{2^{n-1}}) \\ &= \frac{(l_1 + l_2)(l_3^{n-1} - 2^{n-1})}{l_3 - 2} + l_1 l_3^{n-1} + 3^{n-1}. \end{aligned}$$

Now by (3.2), we obtain

$$\begin{aligned} h(n) &= \left(\sum_{i=1}^{2^n-1} (\mu_i + \alpha_i) \right) + 2^{n-1} \\ &= \frac{(l_1 + l_2)(l_3^{n-1} - 2^{n-1})}{l_3 - 2} + l_1 l_3^{n-1} + 3^{n-1} + 2^{n-1}. \end{aligned} \quad (4.10)$$

Since $\frac{l_3^{n-1} - 2^{n-1}}{l_3 - 2} = \sum_{i=0}^{n-2} l_3^i 2^{n-2-i} \equiv l_3 \pmod{2}$ ($n \geq 3$), then

$$\begin{aligned} h(n) &\equiv \frac{(l_1 + l_2)(l_3^{n-1} - 2^{n-1})}{l_3 - 2} + l_1 l_3 + 1 \\ &\equiv (l_1 + l_2)l_3 + l_1 l_3 + 1 \\ &\equiv 1 + l_2 l_3. \end{aligned}$$

Let $(j)_2 := j_2 j_1$ where $0 \leq j \leq 3$. Then

$$\begin{aligned} S_{4n+j} &= 1 + \sum_{i=1}^m \epsilon_i h(i+2) + j_2 h(2) + j_1 h(1) \\ &\equiv 1 + (1 + l_2 l_3) t_n + j_2 h(2) + j_1 h(1), \end{aligned} \quad (4.11)$$

where t_n is the n -th term of the Thue-Morse sequence. By (4.11), when $1 + l_2 l_3 \equiv 0$, $(S_{4n+j})_{n \geq 0}$ modulo 2 is a constant sequence. When $1 + l_2 l_3 \equiv 1$, $(S_{4n+j})_{n \geq 0}$ modulo 2 is the Thue-Morse sequence up to a coding. \square

Example 3. According to Corollary 1, the sum-free set S (modulo 2) in Example 1 is the Thue-Morse sequence beginning by 1. For the sum-free set S in Example 2, the subsequences $(S_{2n})_{n \geq 0}$ and $(S_{2n+1})_{n \geq 0}$ modulo 2 are both the Thue-Morse sequence beginning by 1.

Corollary 2. Let S be the sum-free set corresponding to the sequences of Cantor type (i.e., the fixed point of $\sigma(l, 0, l+2)$ ($l \geq 2$) beginning by 1).

- (1) When l is odd, then $(S_n)_{n \geq 0}$ modulo 2 is the Thue-Morse sequence $(1 - t_n)_{n \geq 0}$.
- (2) When l is even, then $(S_{2n})_{n \geq 0} \equiv (S_{2n+1})_{n \geq 0}$ modulo 2, and they are both the Thue-Morse sequence $(1 - t_n)_{n \geq 0}$.

Proof. In this case, $l_1 = l, l_2 = 0$ and $l_3 = l + 2$. Thus by (4.10), $h(1) = l + 2 \equiv l$ and

$$h(n) = l^2 + 3l + 3^{n-1} \equiv 1, \quad \forall n > 1.$$

When $l \equiv 1$, by (4.11),

$$\begin{aligned} S_{4n+j} &\equiv 1 + t_n + j_2 + j_1 \\ &\equiv 1 + \left(\sum_{i=1}^m \epsilon_i + j_2 + j_1 \right) \\ &\equiv 1 + t_{4n+j}. \end{aligned}$$

Thus for any $n \geq 0$, $S_n \equiv 1 + t_n$.

When $l \equiv 0$, for $n \geq 0$ and $j = 0, 1$,

$$\begin{aligned} S_{2n+j} &= 1 + \sum_{i=1}^m \epsilon_i h(i+1) + j h(1) \\ &\equiv 1 + \sum_{i=1}^m \epsilon_i \equiv 1 + t_n. \end{aligned}$$

Thus for any $n \geq 0$, $S_{2n} \equiv S_{2n+1} \equiv 1 + t_n$. \square

While μ_n increases fast, the corresponding sum-free set is not complicated. In fact, we have

Proposition 1. *Suppose the sequence $(\mu_n)_{n \geq 1}$ is increasing and*

$$\mu_{n+1} > 2 \sum_{i=1}^n \mu_i. \quad (4.12)$$

Let $S_0 = 1$. Then for any $n \geq 1$,

$$S_n = \sum_{i=1}^n \mu_i + \frac{(n+1)(n+2)}{2}.$$

Proof. By Lemma 1, for every $n \geq 1$, we have $\alpha_{n+1} = \alpha_n + 1$ and

$$\alpha_n = \alpha_{n-1} + 1 = \alpha_{n-2} + 1 + 1 = \cdots = \alpha_1 + (n-1) = n,$$

since $\alpha_1 = 1$. Hence

$$S_n = S_0 + \sum_{i=1}^n \mu_i + \sum_{i=1}^n \alpha_i + n = \sum_{i=1}^n \mu_i + \frac{(n+1)(n+2)}{2}.$$

□

Remark 4. (1) *The growth order of μ_n is larger than $O(3^n)$. In fact, if $\mu_n = 2 \sum_{i=1}^{n-1} \mu_i$ for any $n \geq 3$, then $\mu_n = 3\mu_{n-1}$, and $\mu_2 = 2\mu_1$. Hence $\mu_n = 2 \times 3^{n-2} \mu_1$.*
 (2) *The coefficient 2 in (4.12) is not crucial.*

Remark 5. *Assume $(S_n)_{n \geq 0}$ is a sequence given by $S_n = 1 + \sum_{i=1}^m \epsilon_i h(i)$ where $(n)_2 := \epsilon_m \cdots \epsilon_1$ and $(h(i))_{i \geq 1}$ is a positive integer sequence. If $h(i) \equiv 1$ for $i \geq 0$, then $(S_n)_{n \geq 0}$ modulo 2 is the Thue-Morse sequence. If $h(i) \equiv 0$ for $i \geq 0$, then $S_n \equiv 1$.*

Example 4. *Let $h(i) = 2^i$, then S is sum-free. Moreover, $S = \{1, 3, 5, 7, 9, \dots\}$ and it is periodic of period 2.*

Example 5. *Let $h(i) = (i+1)!$, then S is sum-free. Moreover,*

$$S = \{1, 3, 7, 9, 25, 27, 31, 33, \dots\}$$

and it is periodic of period 4.

5. BASE CHANGING

The mapping $\theta : \Sigma \rightarrow \mathfrak{S}$ is bijective, so it is natural to study the properties of the corresponding zero-one sequences of some sum-free sets. In this section, we will show that the corresponding zero-one sequences of the sum-free sets defined below are automatic.

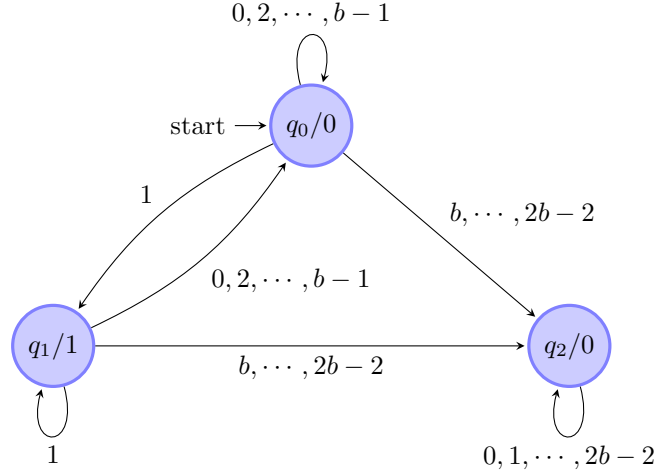
Definition 4. *For any $b \geq 2$, $n \geq 0$, let $S = (S_n)_{n \geq 0}$ be the sum-free set given by $S_n = [(n)_b 1]_{2b-1}$.*

Definition 5 (Allouche and Shallit [3]). *Let \mathcal{A} be a set of non-negative integers. Then we say that \mathcal{A} is a k -automatic set if its characteristic sequence*

$$a_n = \begin{cases} 1, & \text{if } n \in \mathcal{A}; \\ 0, & \text{otherwise,} \end{cases}$$

defines a k -automatic sequence.

Lemma 6. *The set S in Definition 4 is sum-free and $(2b-1)$ -automatic.*

FIGURE 1. Automaton generating the set $(a_n)_{n \geq 0}$ in Lemma 6.

Proof. By the definition of S_n , it is clear that for any integer $n \geq 0$, $(S_n)_{2b-1} \in \Sigma_b^* 1$. Hence, for any integers $m, n \geq 0$, $(S_m + S_n)_{2b-1} \in \Sigma_{2b-1}^* 2$, which implies that $S_m + S_n \notin S$. Thus, the set S is sum-free.

Let $(a_n)_{n \geq 0}$ be the characteristic sequence of S . Then

$$a_n = \begin{cases} 1, & \text{if } (n)_{2b-1} \in \Sigma_b^* 1; \\ 0, & \text{otherwise.} \end{cases}$$

Thus the sequence $(a_n)_{n \geq 0}$ can be generated by the $(2b-1)$ -automaton in Figure 1, which implies that the sequence $(a_n)_{n \geq 0}$ is $(2b-1)$ -automatic. \square

Theorem 4. *The zero-one sequence \mathbf{c} corresponding to the sum-free set S in Definition 4 is $(2b-1)$ -automatic.*

Proof. Recall that the zero-one sequence \mathbf{c} is obtained by deleting all the $*$'s of the sequence $\mathbf{v} = (v_n)_{n \geq 1}$. For any $n \geq 1$, we claim that

- (1) $v_n = 1 \Leftrightarrow (n)_{2b-1} \in \Sigma_b^* 1$,
- (2) $v_n = * \Leftrightarrow (n)_{2b-1} \in \Sigma_{2b-1}^* 2$,
- (3) $v_n = 0 \Leftrightarrow (n)_{2b-1} \in \Sigma_{2b-1}^* \setminus (\Sigma_b^* 1 \cup \Sigma_{2b-1}^* 2)$.

The third assertion is an immediate consequence of the first one and the second one, and the first assertion follows directly from the definitions of S and v_n in (1.1). Therefore it suffices to prove the second assertion.

Since $v_n = * \Leftrightarrow n \in S + S$, we need to show that

$$S + S = (2b-1)\mathbb{N} + 2. \quad (5.1)$$

For any $S_n \in S$, then $S_n = (2b-1)((n)_b)_{2b-1} + 1$. Hence, for any integer $m, n \geq 1$, we have

$$S_m + S_n = (2b-1)((m)_b)_{2b-1} + ((n)_b)_{2b-1} + 2 \in (2b-1)\mathbb{N} + 2.$$

Conversely, for any $n \geq 0$, assume $(n)_{2b-1} := a_k a_{k-1} \cdots a_1$. Hence, for any $1 \leq i \leq k$, there exist $b_i, d_i \in \Sigma_b$ such that $a_i = b_i + d_i$. Thus, there exist two

integers

$$\begin{aligned} n_1 &= [b_k b_{k-1} \cdots b_1]_{2b-1}, \\ n_2 &= [d_k d_{k-1} \cdots d_1]_{2b-1}, \end{aligned}$$

such that $n = n_1 + n_2$ and $(n_1)_{2b-1}, (n_2)_{2b-1} \in \Sigma_b^*$. Moreover,

$$\begin{aligned} (2b-1)n+2 &= ((2b-1)n_1+1) + ((2b-1)n_2+1) \\ &= S_{[b_k b_{k-1} \cdots b_1]_b} + S_{[d_k d_{k-1} \cdots d_1]_b} \in S + S. \end{aligned}$$

This implies (5.1) holds.

Now, we will show that \mathbf{c} is $(2b-1)$ -automatic. By (5.1), $v_{(2b-1)n+2} = *$ for $n \geq 0$. Hence, $\mathbf{c} = c_0 c_1 \cdots$ satisfies

$$\begin{cases} c_{(2b-2)n} &= v_{(2b-1)n+1}; \\ c_{(2b-2)n+i} &= v_{(2b-1)n+i+2}, \end{cases} \quad (5.2)$$

where $1 \leq i \leq 2b-3$. By Lemma 6, S is $(2b-1)$ -automatic. Its characteristic sequence $(a_n)_{n \geq 0}$ is a $(2b-1)$ -automatic sequence. By Theorem 6.8.1 in [3], $(a_{(2b-1)n+i})_{n \geq 0}$ is $(2b-1)$ -automatic for $0 \leq i \leq 2b-2$. Note that $v_n = a_n$ for any $n \notin (2b-1)\mathbb{N} + 2$, then $(v_{(2b-1)n+i})_{n \geq 0}$ is $(2b-1)$ -automatic for $0 \leq i \leq 2b-2$ and $i \neq 2$. Thus $(c_{(2b-2)n+i})_{n \geq 0}$ is $(2b-1)$ -automatic for $0 \leq i \leq 2b-3$. By Theorem 6.8.2 in [3], \mathbf{c} is $(2b-1)$ -automatic. \square

Remark 6. (1) $(2b-1)$ in Theorem 4 cannot be replaced by p , where $p > 2b-1$, since there do not exist $x, y \in \mathbb{N}$ such that $S + S = x\mathbb{N} + y$.
(2) From Theorem 4, if $b = 2$, then the sequence \mathbf{c} is the Cantor sequence

$$101000101000000000101000101 \cdots$$

(3) If we replace $S_n = [(n)_b 1]_{2b-1}$ by $S_n = [(n)_b w]_{2b-1}$ where $w \in \Sigma_b^*$, then the corresponding zero-one sequence is also automatic.

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